Explicit generators for the ring of quasisymmetric functions over the integers

by

Abstract.

MSCS:

Key words and key phrases:

1. The Witt polynomials.

This is well known stuff, see e.g. Chapter 3 of [1], included here for completeness sake and to establish notation.

Let *Symm* be the ring of symmetric functions over the integers in infinitely many variables

$$Symm = \mathbf{Z}[e_1, e_2, \cdots] \subset \mathbf{Z}[x_1, x_2, \cdots] \tag{1.1}$$

Here the e_i are the elementary symmetric functions in the x_j. There is another free polynomial basis of *Symm*, that is related to the free polynomial basis $\{e_1, e_2, \cdots\}$ by the formula

$$\prod_{i=1}^{\infty}(1 - a_i t^i) = 1 - e_1 t + e_2 t^2 - e_3 t^3 + \cdots = \prod_{i=1}^{\infty}(1 - x_i t) \tag{1.2}$$

The free polynomial basis $\{a_1, a_2, \cdots\}$ generalizes in a natural way to a free polynomial basis over the integers for the ring *QSymm* of quasisymmetric functions. It is of course obvious from (1.2) that $\{a_1, a_2, \cdots\}$ is a free polynomial basis of *Symm*.

Let

$$w_n(X) = \sum_{d|n} d X_d^{n/d} \tag{1.3}$$

be the well known Witt polynomials (in a set of commuting variables $X_1, X_2, \cdots$). Let

$$p_n = \sum_i x_i^n \quad \in Symm \tag{1.4}$$

be the power sums. Then

$$w_n(a_1, a_2, \cdots, a_n) = p_n \tag{1.5}$$

To see this just apply $-t\dfrac{d}{dt}\log$ to the formula (1.2) (the outer parts).

2. The wll-ordering

Let $\alpha=[a_1,a_2,a_m]$, $a_i\in\mathbf{N}=\{1,2,\cdots\}$ be a composition. The length of such a composition is $\lg(\alpha)=m$, and its weight is $\mathrm{wt}(\alpha)=a_1+a_2+\cdots+a_m$. The empty composition $[\,]$ has length and weight zero. A composition α defines a monomial quasisymmetric function as follows

$$\alpha=\sum_{i_1<i_2<\cdots<i_m} x_{i_1}^{a_1}x_{i_2}^{a_2}\cdots x_{i_m}^{a_m} \tag{2.1}$$

As a rule we shall not distinguish between a composition and the quasisymmetric function it defines. The empty composition is the unit element in the ring *QSymm* of quasisymmetric functions. The monomial symmetric functions (2.1) form a free Abelian group basis for *QSymm.*

We shall use a total ordering on the set of composition called the wll-ordering. This stands for "weight first, than length, and finally lexicographic". Thus, for instance

$$[5]>_{wll}[1,1,2]>_{wll}[2,2]>_{wll}[1,3]$$

3. Substitution (= plethysm)

Given a compostion α and a composition β define a new quasisymmetric function $\alpha\circ\beta$, "β substituted in α" as follows. Order the summands of the quasisymmetric function β lexicographically and substitute these in that order for the $x_1,x_2,\cdots$ in the quasisymmetric function α. The result is a new quasisymmetric function of weight $\mathrm{wt}(\alpha)\mathrm{wt}(\beta)$.

The transformation

$$s_\beta\colon\ \alpha\mapsto\alpha\circ\beta \tag{3.1}$$

is (obviously) a ring endomorphism. The transformation

$$t_\alpha\colon\ \beta\mapsto\alpha\circ\beta$$

is not a ring homomorphism; it is a plethysm. Indeed, the t_{e_n} define a $\lambda-$ring structure on *QSymm.*

3.2. *Example.* For a composition $\alpha=[a_1,a_2,\cdots a_m]$ and a natural number n let $n\alpha=[na_1,na_2,\cdots,na_m]$[1]. Then for the power sums p_n

$$p_n\circ\alpha=n\alpha \tag{3.3}$$

3.4. *Example.* For two compositions α,β let $\alpha*\beta$ denote their concatenation. Thus, for example, $[1,2]*[1,6,4]=[1,2,1,6,4]$, and $[1,2]^{*3}=[1,2,1,2,1,2]$. Let e_n be the n-th elementary symmetric function. Then if α is a Lyndon word

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[1] It would actually probably be better to write $\mathbf{f}_n\alpha$, for these are the right Frobenius Hopf algebra endomorphisms of *QSymm*; they are also the Adams endomorphisms corresponding to the $\lambda-$ring structure already mentioned.

$$e_n \circ \alpha = \alpha^{*n} + (\text{wll} - \text{smaller}) \tag{3.5}$$

where (wll-smaller) means a sum of monomial quasisymmetric functions that are strictly wll-smaller than α^{*n}.

4. Lyndon-Witt generators

For a composition $\alpha = [a_1, a_2, \cdots, a_m]$ let $g(\alpha) = \gcd\{a_1, a_2, \cdots, a_m\}$ and define

$$A_\alpha = a_{g(\alpha)} \circ \alpha_{red} \tag{4.1}$$

where the a_i are the symmetric functions of section 1 above and

$$\alpha_{red} = [g(\alpha)^{-1} a_1, g(\alpha)^{-1} a_2, \cdots, g(\alpha)^{-1} a_m] \tag{4.2}$$

Note that A_α is homogeneous of weight $\mathrm{wt}(\alpha)$.

4.3. *Lemma.* Let α be a reduced composition, i.e. $g(\alpha) = 1$. Then

$$\sum_{d|n} dA_{d\alpha}^{n/d} = n\alpha \tag{4.4}$$

Proof. This follows immediately from the definition of the A_β by applying the operation "substitute α" to forrmula (1.5), using (3.3).

Note that formula (4.4) again establishes that all A_β are quasisymmetric functions while their integrality is assured by the definition (4.1).

Let *LYN* be the set of Lyndon compositions (Lyndon words).

4.5. *Theorem.* The set $\{A_\alpha\colon \alpha \in LYN\}$ is a set of free polynomial generators for *QSymm*.

Proof. Let R be the subring of *QSymm* generated by the A_α, $\alpha \in LYN$. Because the number of proposed homogeneous generators is just right for each weight it will suffice to show that $R = QSymm$, i.e. that each composition α is in R.

To start with, let β be a Lyndon composition. Then taking $a = \beta_{red}$, and $n = g(\beta)$ in formula (4.4) we se that $\beta \in R$.

We now proceed with induction for the wll-ordering. The case of weight 1 is trivial. For each separate weight the induction starts because of what has just been said because compositions of length 1 are Lyndon.

So let α be a composition of weight ≥ 2 and length ≥ 2. By the Chen-Fox-Lyndon concatenation factorization theorem

$$\alpha = \beta_1^{*r_1} * \beta_2^{*r_2} * \cdots * \beta_k^{*r_k}, \quad \beta_i \in LYN, \quad \beta_1 >_{lex} \beta_2 >_{lex} \cdots >_{lex} \beta_k \tag{4.6}$$

where, as before, the $*$ denotes concatenation and $\beta >_{lex} \beta'$ means that β is lexicographically strictly larger than β'.

If $k \geq 2$, take $\alpha' = \beta_1^{*r_1}$ and for α'' the corresponding tail of α so that $\alpha = \alpha' * \alpha''$. Then

$$\alpha'\alpha'' = \alpha' * \alpha'' + (\text{wll} - \text{smaller than } \alpha) = \alpha + (\text{wll} - \text{smaller than } \alpha)$$

and with induction it follows that $\alpha \in R$. there remains the case that $k = 1$ in the CFL-factorization (4.6). If then also $r = r_1 = 1$, α is Lyndon, and hence by what has been said at the start of the proof $\alpha \in R$. The remaining case is that of a word of the form $\gamma = \beta^{*r}$, $r \geq 2$ and this case requires a rather different argument.

Let $\mathcal{P}$ be the ideal in *QSymm* generated by all nontrivial products $\alpha_1\alpha_2$, $\lg(\alpha_1), \lg(\alpha_2) \geq 1$. Now in formula (4.4) take $\alpha = \beta_{red}$ and $n = g(\beta)r$ to see that

$$g(\beta)rA_{r\beta} = g(\beta)rA_{g(\beta)r\alpha} \equiv g(\beta)r\alpha = r\beta \mod \mathcal{P} \tag{4.7}$$

Now consider the r-th Newton relation in *Symm*

$$p_r - e_1 p_{r-1} + \cdots + (-1)^{r-1} e_{r-1} p_1 + (-1)^r r e_r = 0 \tag{4.8}$$

And apply the operation 'substitute β ' to it. Using Examples 3.2, 3.4, there results that

$$r\beta \equiv \pm r(\beta^{*r} + (\text{wll - smaller than } \beta^{*r})) \mod \mathcal{P} \tag{4.9}$$

Now combine this with (4.7) and use that $QSymm / \mathcal{P}$ is torsion free to see that (for Lyndon β)

$$g(\beta)A_{r\beta} \equiv \pm \beta^{*r} + (\text{wll - smaller than } \beta^{*r}) \tag{4.10}$$

With induction this finishes the proof.

4.11. *Remarks.*

Note that this proof requires that one has already shown that $QSymm / \mathcal{P}$ is torsion free which follows from the theorem that abstractly (without specifying explicit generators) the ring *QSymm* is freely polynomially generated, which is proved in [3, 4].

The idea of using Chen-Fox-Lyndon factorization to prove theorems like 4.5 goes back (at least) to [5]. This is also the key technique for proving a p-adic version of theorem 4.5, i.e. over $\mathbf{Z}_{(p)}$, (also with explicit generators) in [2, 3]. Which, in turn, suffices to establish the torsion freeness of $QSymm / \mathcal{P}$.

4.12. *Corollary*

Take another free polynomial basis of *Symm*, like the elementary symmetric functions e_n or the complete symmetric functions h_n. Define quasisymmetric functions

$$E_\alpha = e_{g(\alpha)} \circ \alpha_{red}, \quad H_\alpha = h_{g(\alpha)} \circ a_{red} \tag{4.13}$$

Then also $\{E_\alpha \colon \alpha \in LYN\}$ and $\{H_\alpha \colon \alpha \in LYN\}$ are free polynomial bases for *QSymm*.

References.

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